Math/Music: Structure and Form
Strähle's Guitar Construction

In 1743, Swedish craftsman Daniel Strähle published an article in the *Proceedings of the Swedish Academy* describing a simple and accurate method for placing the frets on a guitar. In fact, Strähle's method works for any stringed instrument with a fingerboard (violin, viola, lute, ukulele, etc.). Unfortunately, the geometer and economist Jacob Faggot studied Strähle's method in detail and concluded that it contained errors of up to 1.7% when measured against Equal Temperament. Since this was a large enough error to be noticed by a musician, the method was discarded, particularly because Faggot was a founding member of the Swedish Academy, and was once ranked fourth by the Academy. However, Faggot had made a tragic error, as was discovered much later by mathematician J. M. Barbour (Michigan State University) in 1957. It turns out that Strähle's method is remarkably accurate, as demonstrated below.

Strähle's method begins with a particular guitar and an isosceles triangle with congruent side lengths of 24 and a base of length 12. Place the guitar so that the neck sits at one vertex of the triangle and half the string length sits 7 units up the opposite side of the triangle (see Figure 1). Next, draw 12 lines from the top of the triangle to equally spaced intervals along the base. The intersection of these 12 lines with the fretboard gives the location of the frets. That's it!

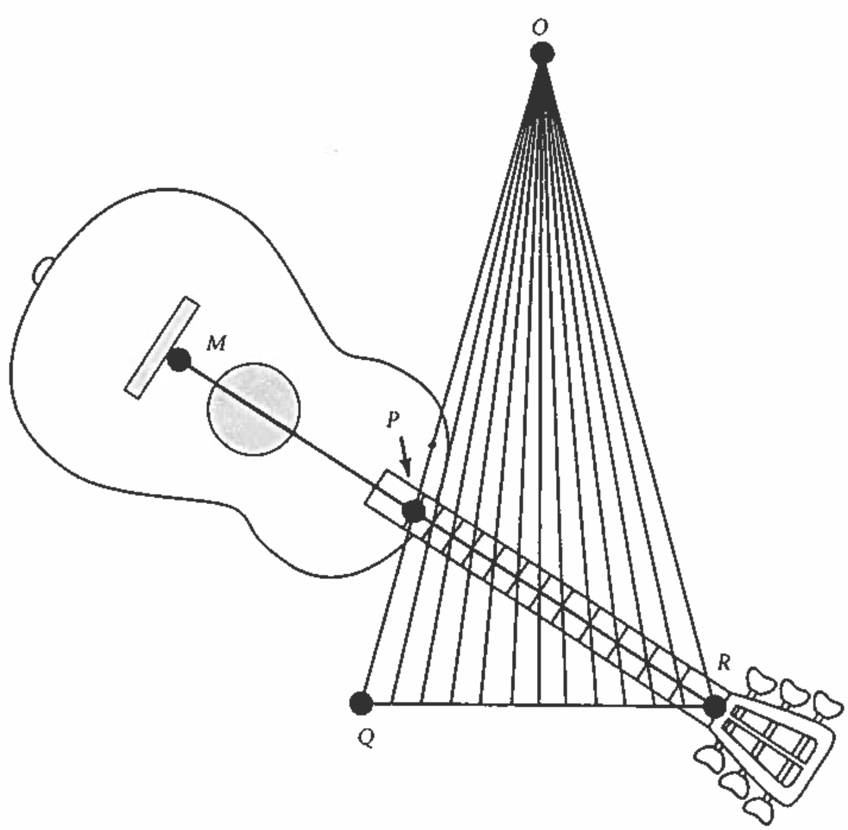

Figure 1: Strähle's construction to place the frets. Sides OQ and OR are length 24, while the base QR is 12 units. P is the midpoint of the guitar string and PQ is 7 units long.

By using similar triangles, it is possible to use a similar method for any size string instrument (see Figure 2).

Figure 6. Strähle's illustration of the practical application of his method. Lay the fingerboard parallel to the line *RPM* and adjust to length, with the midpoint on the line *OW*; now mark the frets.

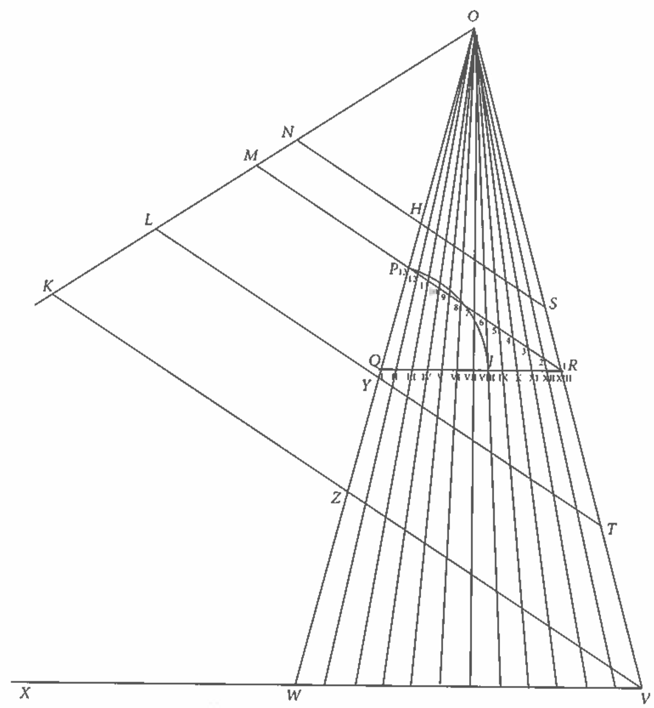

Figure 2: Extending Strähle's method to any size instrument. Since fret placement is based on ratios (just like frequency multipliers with the different tuning systems we have studied), the method scales to different size instruments using similar triangles.

Strähle's placement of the frets on the guitar corresponds to the linear fractional transformation

$$y = \frac{17 - 5x}{17 + 7x}.$$

Here, we break the base of the isosceles triangle (the x-axis) into 12 equal parts and label the rightmost point $x = 0$ (the point R in Figure 1) and the leftmost point $x = 1$ (the point Q in Figure 1). The neck of the guitar containing the frets is the y-axis with the point R labeled $y = 1$ and the point M (where the string is fastened to the base of the guitar) is labeled $y = 0$. This means that $y = 1/2$ is at the point P, the midpoint of the string. Consequently, placing your finger at P and plucking the string yields a note an octave higher than the fundamental note of the full string. Thus, the value of y is precisely the ratio between the lengths of the original string and the plucked string when placing a finger down on the fret at point y. Moreover, since we have subdivided the base of the triangle into 12 equal parts, with $(x = 0, y = 1)$ corresponding to the fundamental note and $(x = 1, y = 1/2)$ corresponding to an octave higher, the value of y for each x-value of the form $x = n/12$, gives Strähle's frequency ratio for moving up n half steps in pitch.

For example, $y = 1$ (point R), means pluck the original string, so the frequency ratio is $1/1$ or $y = 1$. In class, we showed that $y = 29/41$ is the value where the altitude of the isosceles triangle intercepts the guitar string. Therefore, setting $n = 6$, we have that $x = 1/2$ maps to $y = 29/41$ (check it in the formula above) and thus 6 half steps (the tritone) is a ratio of 29/41. This is an excellent approximation to the ratio obtained in Equal Temperament:

$$\text{6 half steps} \;=\; 2^{6/12} \;=\; 2^{1/2} \;=\; \sqrt{2}.$$

However, since we are comparing lengths of strings and this is the *inverse* of frequency, we really want to compare with $1/\sqrt{2}$. Therefore, if Strähle's method is to make any sense at all, we would need 29/41 to approximate $1/\sqrt{2}$. Inverting each fraction, this is equivalent to

$$\frac{41}{29} \approx 1.4138 \quad \text{approximates} \quad \sqrt{2} \approx 1.4142,$$

an excellent approximation!

We have seen the number 41/29 before. It is the fifth convergent p_4/q_4 in the continued fraction expansion of $\sqrt{2}$ (see the top of page 4 on the *Continued Fractions* handout). Recall that the convergents obtained from the continued fraction expansion of an irrational number are the *best* approximating rational numbers with small denominators. The fact that Strähle's method just happens to use a convergent from the continued fraction expansion for $\sqrt{2}$ is one of the primary reasons his method is so effective.

Table 1 compares the ratios for Strähle's method with those of Equal Temperament. Note that the maximum total error is only 0.152%. Also note the rather large values in the fractions. Table 2 shows a comparison of the inverse of these values (frequency ratios) in terms of cents, with Just Intonation included. Remarkably, the discrepancy between Strähle's method and Equal Temperament is never more than 3 cents.

Note	Interval	Strähle	Equal Temp.	% Error
C	unison	$\frac{1}{1}$	1	0
C♯	minor second	$\frac{199}{211} \approx 0.9431$	$2^{-1/12} \approx 0.9439$	-0.079
D	major second	$\frac{97}{109} \approx 0.8899$	$2^{-1/6} \approx 0.8909$	-0.111
D♯	minor third	$\frac{21}{25} = 0.84$	$2^{-1/4} \approx 0.8409$	-0.107
E	major third	$\frac{23}{29} \approx 0.7931$	$2^{-1/3} \approx 0.7937$	-0.075
F	perfect fourth	$\frac{179}{239} \approx 0.7490$	$2^{-5/12} \approx 0.7492$	-0.027
F♯	tritone	$\frac{29}{41} \approx 0.7073$	$2^{-1/2} \approx 0.7071$	0.030
G	perfect fifth	$\frac{169}{253} \approx 0.6680$	$2^{-7/12} \approx 0.6674$	0.085
G♯	minor sixth	$\frac{41}{65} \approx 0.6308$	$2^{-2/3} \approx 0.6300$	0.128
A	major sixth	$\frac{53}{89} \approx 0.5955$	$2^{-3/4} \approx 0.5946$	0.152
A♯	minor seventh	$\frac{77}{137} \approx 0.5620$	$2^{-5/6} \approx 0.5612$	0.145
B	major seventh	$\frac{149}{281} \approx 0.5302$	$2^{-11/12} \approx 0.5297$	0.098
C	octave	$\frac{1}{2}$	$\frac{1}{2}$	0

Table 1: Comparing the ratios of Strähle's method and Equal Temperament. Note that the largest total error is only 0.152%.

Note	Interval	Strähle	Equal Temp.	Just Int.
C	unison	0	0	0
C♯	minor second	101.37	100	111.7
D	major second	201.9	200	203.9
D♯	minor third	301.8	300	315.6
E	major third	401.3	400	386.3
F	perfect fourth	500.5	500	498.0
F♯	tritone	599.5	600	590.2
G	perfect fifth	698.5	700	702.0
G♯	minor sixth	797.8	800	813.7
A	major sixth	897.4	900	884.4
A♯	minor seventh	997.5	1000	996.1
B	major seventh	1098.3	1100	1088.3
C	octave	1200	1200	1200

Table 2: Comparing the frequency ratios (multipliers) of Strähle's method, Equal Temperament and Just Intonation in terms of cents. Note that Strähle's method is never more than 3 cents away from Equal Temperament.